

On prime spectra of Leavitt path algebras

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Abstract

We associate in a natural way to any partially ordered set (P, \leq) a directed graph E_P (where the vertices of E_P correspond to the elements of P , and the edges of E_P correspond to related pairs of elements of P), and then describe the prime spectrum $\text{Spec}(L_K(E_P))$ of the resulting Leavitt path algebra $L_K(E_P)$. This construction allows us to realize a wide class of partially ordered sets as the prime spectra of rings. More specifically, any partially ordered set in which every downward directed subset has a greatest lower bound, and where these greatest lower bounds satisfy certain compatibility conditions, can be so realized. In particular, any partially ordered set satisfying the descending chain condition is in this class.

Keywords: prime spectrum, partially ordered set, Leavitt path algebra

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For a ring R , the partially ordered set $(\text{Spec}(R), \subseteq)$, consisting of the prime ideals of R under set inclusion, has been the focus of significant research attention for many decades. While classically such investigations dealt only with commutative unital rings, recent energy has been spent studying more general classes. Specifically, one may ask whether there are any necessary conditions on the poset $(\text{Spec}(R), \subseteq)$ for general R . More precisely, one may pose a “Realization Question”: given a partially ordered set (P, \leq) , does there exist a ring R for which $(\text{Spec}(R), \subseteq) \cong (P, \leq)$? In this article we answer such a question in the affirmative for a large class of partially ordered sets, thereby extending the class for which an affirmative answer was heretofore known. In the process, we are led to a naturally occurring collection of partially ordered sets, an until-now-unidentified collection which seems to be of interest in its own right. The class of algebras by which we realize these posets as prime spectra are built as Leavitt path algebras. Such algebras also arise in this Realization Question context, as they have at their heart a directed graph, and there is a natural way (germane to the Realization Question, and described herein) to associate a directed graph E_P to any given poset P .

The article is organized as follows. We start by reminding the reader of various definitions and properties of partially ordered sets, as well as of the definition of Leavitt path algebras. After providing a necessary general condition on $\text{Spec}(R)$ for arbitrary R , which will give some context to our results (Property GLB, Proposition 3), we present the definition of the directed graph E_P arising from a partially ordered set P . Subsequently, we analyze the relationship between (P, \leq) and $(\text{Spec}(L_K(E_P)), \subseteq)$, where $L_K(E_P)$ denotes the Leavitt path

algebra of E_P with coefficients in a field K . We show (Proposition 13) that the descending chain condition on P is necessary and sufficient in order that the easily-anticipated map from P to $\text{Spec}(L_K(E_P))$ is an order-isomorphism. In our main Realization Question result (Theorem 18), we show that $\text{Spec}(L_K(E_P))$ is isomorphic to a partially ordered set $\mathcal{A}(P)$ built from P by appropriately adding greatest lower bounds to certain subsets of P . We then proceed to show (Theorem 27) that the construction of the poset $\mathcal{A}(P)$ from P may be interpreted quite naturally in purely poset-theoretic terms. Specifically, for any poset P we define the poset $\mathcal{R}(P)$, by removing greatest lower bounds from certain subsets of P , and then show that $P \cong \mathcal{A}(Q)$ for some poset Q precisely when $\mathcal{R}(\mathcal{A}(P)) \cong \mathcal{A}(\mathcal{R}(P))$. We also provide in Theorem 27 three conditions on a poset P which, taken together, are necessary and sufficient to ensure that $P \cong \mathcal{A}(Q)$. The remainder of the article is spent showing how various of these three conditions play out in other contexts. In particular, we show how these conditions relate to a currently open question (motivated by a classical result of Kaplansky) regarding the structure of the prime spectrum of an arbitrary ring.

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1 Partially ordered sets, directed graphs, and Leavitt path algebras

In this initial section we give an overview of the key foundational ideas used in this article.

Partially ordered sets

A *preordered set* (P, \leq) is a set P together with a binary relation \leq which is reflexive and transitive. If in addition \leq is antisymmetric, then (P, \leq) is called a *partially ordered set* (often shortened simply to *poset*). If a poset (P, \leq) has the property that $p \leq q$ or $q \leq p$ for all $p, q \in P$, then (P, \leq) is called a *totally ordered set*, or alternatively, a *chain*.

Let (P, \leq) be a partially ordered set. An element x of P is called

- a *minimal* element of P if there is no $y \in P$ such that $y < x$, and
- a *least* element of P if $x \leq y$ for all $y \in P$.

Clearly if P contains a least element then it is necessarily unique. P is called an *infinite descending chain* if P is totally ordered, and P has no minimal elements (if and only if P has no least element). P is *downward directed* if it is nonempty, and for all $p, q \in P$ there exists $r \in P$ such that $p \geq r$ and $q \geq r$. If P is downward directed and contains a minimal element, then it is necessarily the least element of P .

A *Hasse diagram* of (P, \leq) is a diagram (if one exists) representing the elements of the set P as points, with a line drawn upwards from a point x to a point y whenever $x < y$ and there is no element $z \in P$ satisfying $x < z < y$.

Let (P_1, \leq_1) and (P_2, \leq_2) be two partially ordered sets, and let $f : P_1 \rightarrow P_2$ be a function. Then f is *order-preserving* if for all $x, y \in P_1$, $x \leq_1 y$ implies that $f(x) \leq_2 f(y)$. Also f is *order-reflecting* if for all $x, y \in P_1$, $f(x) \leq_2 f(y)$ implies that $x \leq_1 y$. In this case, f is necessarily injective, since $f(x) = f(y)$ implies that $x \leq_1 y$ and $y \leq_1 x$. If f is both order-preserving and order-reflecting, then it is an *order-embedding*. If f is an order-embedding and bijective, then it is an *order-isomorphism*. We write $(P_1, \leq_1) \cong (P_2, \leq_2)$ if there is an order-isomorphism between the two partially ordered sets.

For any poset (P, \leq) and subset S of P , by restriction (S, \leq) is a poset. We say that (P, \leq) satisfies the *descending chain condition* (shortened as *DCC*) if there is no subset S of P for which (S, \leq) is an infinite descending chain. A *lower bound* for $S \subseteq P$ is an element x of P such that $x \leq s$ for all $s \in S$. A *greatest lower bound* of S is a lower bound x of S with the additional property that $y \leq x$ for every $y \in P$ having $y \leq s$ for all $s \in S$. Easily if such a greatest lower bound x of S is an element of S itself, then x is the least element of S . Just as easily, if a greatest lower bound for S exists, it is necessarily unique. If (P, \subseteq) is a set of sets ordered by inclusion, and $S \subseteq P$ is a collection for which $x = \bigcap_{s \in S} s \in P$, then necessarily x is the greatest lower bound of S .

The following three subsets of \mathbb{Z} (the partially ordered set of integers) will be of importance to us: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$, and $\mathbb{Z}^- = \{-1, -2, -3, \dots\}$.

Remark 1. If a poset P has DCC, then easily every downward directed subset S of P (in particular, every chain in P) necessarily has a greatest lower bound in P (indeed, has a least element in S). However, there are posets P where every downward directed subset has a greatest lower bound in P , but which do not have DCC. The poset $P = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ with the usual order is such. \square

Directed graphs

A *directed graph* $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$ consists of two sets E^0, E^1 (the elements of which are called *vertices* and *edges*, respectively), together with functions $\mathbf{s}, \mathbf{r} : E^1 \rightarrow E^0$, called *source* and *range*, respectively. We shall refer to directed graphs as simply “graphs” from now on. A *path* p in E is a finite sequence $e_1 \cdots e_n$ of edges $e_1, \dots, e_n \in E^1$ such that $\mathbf{r}(e_i) = \mathbf{s}(e_{i+1})$ for $i \in \{1, \dots, n-1\}$. Here we define $\mathbf{s}(p) := \mathbf{s}(e_1)$ to be the *source* of p and $\mathbf{r}(p) := \mathbf{r}(e_n)$ to be the *range* of p . We view the elements of E^0 as paths of length 0, and denote by $\text{Path}(E)$ the set of all paths in E . A path $p = e_1 \cdots e_n$ is said to be *closed* if $\mathbf{s}(p) = \mathbf{r}(p)$. Such a path is said to be a *cycle* if the path has nonzero length, and in addition $\mathbf{s}(e_i) \neq \mathbf{s}(e_j)$ for every $i \neq j$. A cycle consisting of just one edge is called a *loop*. A graph which contains no cycles is called *acyclic*.

A vertex $v \in E^0$ for which the set $\mathbf{s}^{-1}(v) = \{e \in E^1 \mid \mathbf{s}(e) = v\}$ is finite is said to have *finite out-degree*. A graph E is said to have *finite out-degree*, or to be *row-finite*, if every vertex of E has finite out-degree. A vertex $v \in E^0$ such that $\mathbf{s}^{-1}(v) = \emptyset$ is called a *sink*, while a vertex having finite out-degree which is not a sink is called *regular*. If $u, v \in E^0$ are distinct vertices such there is a path $p \in \text{Path}(E)$ satisfying $\mathbf{s}(p) = u$ and $\mathbf{r}(p) = v$, then we write $u > v$. It is easy to see that (E^0, \geq) is a preordered set. Given a vertex $v \in E^0$, set

$\mathcal{M}(v) = \{w \in E^0 \mid w \geq v\}$. A subset $H \subseteq E^0$ is *hereditary* if whenever $u \in H$ and $u \geq v$ for some $v \in E^0$, then $v \in H$. Also $H \subseteq E^0$ is *saturated* if $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H$ implies that $v \in H$ for any regular $v \in E^0$. A nonempty subset $M \subseteq E^0$ is a *maximal tail* if it satisfies the following conditions.

(MT1) If $v \in M$ and $u \in E^0$ are such that $u \geq v$, then $u \in M$.

(MT2) For every regular $v \in M$ there exists $e \in E^1$ such that $\mathbf{s}(e) = v$ and $\mathbf{r}(e) \in M$.

(MT3) For all $u, v \in M$ there exists $w \in M$ such that $u \geq w$ and $v \geq w$.

For any subset $H \subseteq E^0$ it is easy to see that H is hereditary if and only if $M = E^0 \setminus H$ satisfies MT1, and H is saturated if and only if $M = E^0 \setminus H$ satisfies MT2.

Leavitt path algebras

For any field K and (nonempty) graph E there are a number of well-studied ways in which one may produce an associative K -algebra which reflects the structure of E . In the current context, the Leavitt path algebra provides us with an effective tool. Precisely, the *Leavitt path K -algebra* $L_K(E)$ of E is the K -algebra generated by the set $\{v \mid v \in E^0\} \cup \{e, e^* \mid e \in E^1\}$, subject to the following relations:

(V) $vw = \delta_{v,w}v$ for all $v, w \in E^0$,

(E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$ for all $e \in E^1$,

(E2) $\mathbf{r}(e)e^* = e^*\mathbf{s}(e) = e^*$ for all $e \in E^1$,

(CK1) $e^*f = \delta_{e,f}\mathbf{r}(e)$ for all $e, f \in E^1$, and

(CK2) $v = \sum_{e \in \mathbf{s}^{-1}(v)} ee^*$ for every regular vertex $v \in E^0$.

The Leavitt path algebra $L_K(E)$ may also be viewed as follows. Given a (directed) graph E , let \widehat{E} denote the directed graph for which $\widehat{E}^0 = E^0$ and $\widehat{E}^1 = E^1 \sqcup (E^1)^*$, where $(E^1)^* = \{e^* \mid e \in E^1\}$ and for each $e^* \in (E^1)^*$, $\mathbf{s}(e^*) = \mathbf{r}(e)$ and $\mathbf{r}(e^*) = \mathbf{s}(e)$. (Throughout, \sqcup denotes “disjoint union.”) That is, \widehat{E} is constructed from E by adding to E^1 , for each edge e , a new edge e^* having orientation opposite to that of e . Then $L_K(E)$ is the quotient of the standard path algebra $K\widehat{E}$, modulo the relations CK1 and CK2.

For all $v \in E^0$ we define $v^* := v$, and for all paths $\alpha = e_1 \cdots e_n$ ($e_1, \dots, e_n \in E^1$) we set $\alpha^* := e_n^* \cdots e_1^*$, $\mathbf{r}(\alpha^*) := \mathbf{s}(\alpha)$, and $\mathbf{s}(\alpha^*) := \mathbf{r}(\alpha)$. With this notation, every element of $L_K(E)$ can be expressed (though not necessarily uniquely) in the form $\sum_{i=1}^n k_i \alpha_i \beta_i^*$ for some $k_i \in K$ and $\alpha_i, \beta_i \in \text{Path}(E)$.

It is easily established that $L_K(E)$ is unital (with multiplicative identity $\sum_{v \in E^0} v$) if and only if E^0 is a finite set.

For additional information and background on Leavitt path algebras see e.g., [1].

2 Prime ideals

Recall that given a (not necessarily unital) ring R , a proper ideal $I \subseteq R$ is *prime* if for all $x, y \in R$, $xRy \subseteq I$ implies that either $x \in I$ or $y \in I$ (equivalently, if for all ideals $A, B \subseteq R$, $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$). The set of prime ideals of R , denoted $\text{Spec}(R)$, is clearly is a poset under set inclusion. In [6, Theorems 9 and 11] Kaplansky considers the question of which partially ordered sets can arise as $\text{Spec}(R)$, where R is a commutative unital ring. We summarize those results here.

Theorem 2 (Kaplansky). *Let R be a commutative unital ring.*

- (1) *If $\{P_i \mid i \in I\}$ is a totally ordered set of prime ideals of R , then $\bigcap_{i \in I} P_i$ is a prime ideal of R . (Rephrased: every totally ordered subset of $\text{Spec}(R)$ has a greatest lower bound in $\text{Spec}(R)$.)*
- (2) *If $\{P_i \mid i \in I\}$ is a totally ordered set of prime ideals of R , then $\bigcup_{i \in I} P_i$ is a prime ideal of R .*
- (3) *If $P \subset Q$ are distinct prime ideals of R , then there exist distinct prime ideals P' and Q' of R such that $P \subseteq P' \subset Q' \subseteq Q$, and there are no prime ideals properly between P' and Q' .*

Theorem 2 can be established nearly verbatim, with essentially the same proofs as given in [6], for not-necessarily-unital commutative rings. (Statement (2) must be modified to include the possibility in this generality that the union of a totally ordered set of prime ideals might yield the non-prime ideal R of R .) The three obvious questions regarding the generalizations of the three assertions of Theorem 2 to noncommutative rings yield three different answers.

Statement (1) extends verbatim (and more generally): see Proposition 3.

Statement (2) does not extend: for instance, Bergman has constructed a noncommutative (unital) ring having a chain of prime ideals whose union is not prime (see [9, Example 4.2] or [4, Example 2.1]). A Leavitt path algebra with this property is given in Example 6 below.

Statement (3): whether or not Statement (3) extends to all rings is currently not known. (We discuss this question in Section 5; a nice account of various aspects of this question is given in [11].)

Here is the extension of Statement (1) of Theorem 2 to noncommutative rings.

Proposition 3. *Let R be any ring, and let $\{P_i \mid i \in I\}$ be a downward directed collection of prime ideals of R ; that is, for all $i, j \in I$ there exists $k \in I$ such that $P_k \subseteq P_i \cap P_j$. Then $P = \bigcap_{i \in I} P_i$ is a prime ideal. Consequently, if (P, \leq) is a partially ordered set that can be represented as the prime spectrum of a ring, then every downward directed subset of P has a greatest lower bound.*

Proof. Since all the P_i are proper ideals of R , so is P . Now, let $x, y \in R$ be any elements, and suppose that $xRy \subseteq P$ but $x \notin P$. Then $x \notin P_i$ for some $i \in I$, and hence $y \in P_i$, as P_i is prime. By hypothesis, for any $j \in I$ there exists $k \in I$ such that $P_k \subseteq P_i \cap P_j$. Since $P_k \subseteq P_i$, we have $x \notin P_k$ but $xRy \subseteq P_k$. Thus $y \in P_k$, and hence also $y \in P_j$. Since $j \in I$ was arbitrary, we conclude that $y \in P$, and therefore P is prime.

The final claim follows from the fact that $\bigcap_{i \in I} P_i$ is necessarily the greatest lower bound in $\text{Spec}(R)$ of the downward directed set $\{P_i \mid i \in I\}$. \square

We will denote by GLB the property of $\text{Spec}(R)$ described in Proposition 3, specifically that every downward directed subset has a greatest lower bound.

The structure of $\text{Spec}(R)$ for commutative unital R , and a discussion of the Realization Question, is given in various places, including [5, 7, 8, 12].

The structure of $\text{Spec}(L_K(E))$ for a finite graph E was described in [3]. Subsequently, an explicit description of the prime ideals of $L_K(E)$ for an arbitrary graph E was given by Rangaswamy in [10, Theorem 3.12]. We present this result now, once a modicum of additional notation has been established. In a graph E , a cycle $c \in \text{Path}(E)$ is said to be *WK* (for “without Condition (K)”) if no vertex along c is the source of another distinct cycle in E (i.e., one possessing a different set of edges). A vertex $v \in E^0$ is called a *breaking vertex* of a hereditary saturated set $H \subseteq E^0$ if $v \in E^0 \setminus H$, $|\mathbf{s}^{-1}(v)| \geq \aleph_0$, and $1 \leq |\mathbf{s}^{-1}(v) \cap \mathbf{r}^{-1}(E^0 \setminus H)| < \aleph_0$. The set of all breaking vertices of H is denoted by B_H . Given $v \in B_H$, we define $v^H \in L_K(E)$ by setting $v^H = v - \sum_{\mathbf{s}(e)=v, \mathbf{r}(e) \notin H} ee^*$.

Theorem 4 (Rangaswamy). *Let E be a graph, K a field, I a proper ideal of $L_K(E)$, and $H = I \cap E^0$. Then I is a prime ideal if and only if I satisfies one of the following conditions.*

- (1) $I = \langle H \cup \{v^H \mid v \in B_H\} \rangle$ (where B_H may be empty) and $E^0 \setminus H$ satisfies MT3.
- (2) $I = \langle H \cup \{v^H \mid v \in B_H \setminus \{u\}\} \rangle$ for some $u \in B_H$ and $E^0 \setminus H = \mathcal{M}(u)$.
- (3) $I = \langle H \cup \{v^H \mid v \in B_H\} \cup \{f(c)\} \rangle$ where $c \in \text{Path}(E) \setminus E^0$ is a WK cycle having source $u \in E^0$, $E^0 \setminus H = \mathcal{M}(u)$, and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.

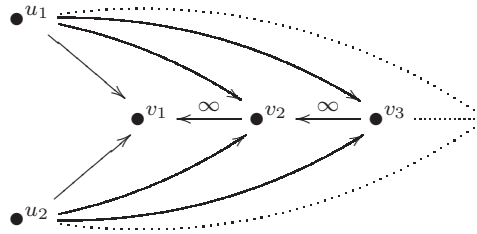
Remark 5. It is easy to see that $H = I \cap E^0$ is hereditary and saturated for any ideal $I \subseteq L_K(E)$. Also, it is noted in [10, Section 2] that

$$\langle H \cup \{v^H \mid v \in B_H\} \rangle \cap E^0 = H$$

for any hereditary and saturated subset H of E^0 . □

Theorem 4 provides us with all the information we need to construct a Leavitt path algebra containing a chain of prime ideals whose union is not prime.

Example 6. Let K be any field, and let E be the following graph.



(The symbol ∞ appearing adjacent to an edge indicates that there are countably infinitely many edges from one vertex to the other.) Then the two sets $\{u_1\}$ and $\{u_2\}$ are maximal tails of E , and all the other maximal tails of E are of the form $M_n = \{u_1, u_2\} \cup \{v_i \mid i \geq n\}$ ($n \in \mathbb{Z}^+$).

In particular, it follows from Theorem 4(1) that $\langle E^0 \setminus M_n \rangle = \langle \{v_1, \dots, v_{n-1}\} \rangle$ is a prime ideal in $L_K(E)$ for each $n \geq 1$ (where $\langle E^0 \setminus M_1 \rangle = 0$). However,

$$\bigcup_{i=1}^{\infty} \langle E^0 \setminus M_i \rangle = \langle \{v_i \mid i \geq 1\} \rangle$$

is not a prime ideal, by Theorem 4, since $E^0 \setminus \{v_i \mid i \geq 1\} = \{u_1, u_2\}$ does not satisfy MT3. (Note that $E^0 \cap \langle \{v_i \mid i \geq 1\} \rangle = \{v_i \mid i \geq 1\}$ by Remark 5.) □

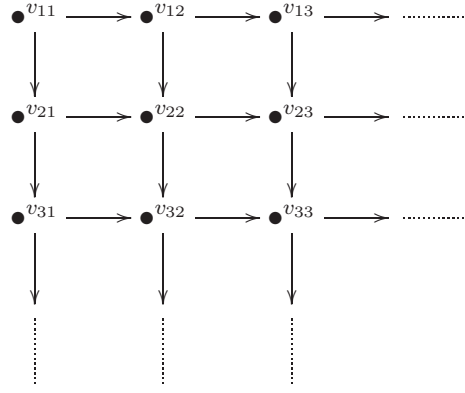
We mention in passing that it can be shown that adjoining a unit to the ring $L_K(E)$ given in the previous example produces a unital ring with a chain of prime ideals whose union is not prime.

Corollary 7. *Let K be a field, and let E be an acyclic graph such that $B_H = \emptyset$ for every hereditary saturated set $H \subseteq E^0$. Then an ideal I of $L_K(E)$ is prime if and only if it is of the form $I = \langle H \rangle$, where H is a proper subset of E^0 such that $E^0 \setminus H$ is a maximal tail.*

Proof. If I is a prime ideal, then, by Theorem 4, $I = \langle I \cap E^0 \rangle$, since E is acyclic and $B_{I \cap E^0} = \emptyset$, where $E^0 \setminus (I \cap E^0)$ satisfies MT3. Since I is an ideal, $I \cap E^0$ must be hereditary and saturated, Remark 5. From this it follows that $E^0 \setminus (I \cap E^0)$ satisfies MT1 and MT2, and is hence a maximal tail. Finally, $I \cap E^0 \neq E^0$, since $\langle E^0 \rangle = L_K(E)$, and I is prime.

Conversely, suppose that $H \subset E^0$ is such that $E^0 \setminus H$ satisfies MT1, MT2, and MT3. Then, as mentioned above, H must be hereditary and saturated. Hence, by hypothesis, $B_H = \emptyset$. Moreover, by Remark 5, $\langle H \rangle$ must be a proper ideal of $L_K(E)$. Thus $I = \langle H \rangle$ is prime, by Theorem 4. \square

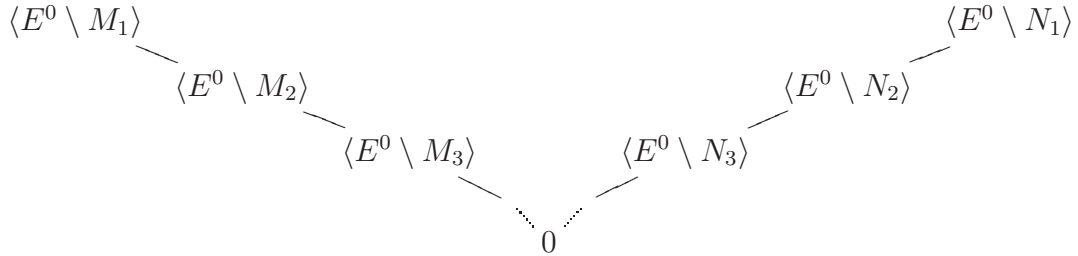
Example 8. Let K be any field, and let E be the graph pictured here.



Then the following are all the maximal tails of E :

$$M_n = \{v_{ij} \mid i \leq n\} \ (n \in \mathbb{Z}^+), \quad N_n = \{v_{ij} \mid j \leq n\} \ (n \in \mathbb{Z}^+), \quad \text{and} \quad E^0.$$

That is, M_n consists of the vertices in the first n rows of E , and N_n consists of the vertices in the first n columns of E . Clearly E is acyclic, and $B_H = \emptyset$ for every hereditary saturated set $H \subseteq E^0$ (since E has only regular vertices). Thus, by Corollary 7, the prime spectrum of $L_K(E)$ has the following Hasse diagram.



We will revisit this example in Section 4. \square

The following lemma will be useful in the sequel.

Lemma 9. *Let E be a graph, $\{M_i \mid i \in I\}$ a collection of maximal tails of E^0 , and $M = \bigcup_{i \in I} M_i$. Then the following hold.*

- (1) M satisfies MT1 and MT2.
- (2) M satisfies MT3 if and only if for all $u, v \in M$ there exists $i \in I$ such that $u, v \in M_i$.

Proof. (1) Suppose that $v \in M$ and $u \in E^0$ are such that $u \geq v$. Then there is some $i \in I$ such that $v \in M_i$. Since M_i satisfies MT1, we have $u \in M_i$, and hence $u \in M$, showing that M satisfies MT1.

Next, let $v \in M$ be a regular vertex, and let $i \in I$ be such that $v \in M_i$. Since M_i satisfies MT2, there exists $e \in E^1$ such that $\mathbf{s}(e) = v$ and $\mathbf{r}(e) \in M_i$. Hence $\mathbf{r}(e) \in M$, and therefore M satisfies MT2.

(2) Suppose that for all $u, v \in M$ there exists $i \in I$ such that $u, v \in M_i$. Let $u, v \in M$ and let $i \in I$ be such that $u, v \in M_i$. Since M_i satisfies MT3, there exists $w \in M_i$ such that $u \geq w$ and $v \geq w$. Since $w \in M$, it follows that M satisfies MT3.

Conversely, suppose that M satisfies MT3, and let $u, v \in M$. Then there exists $w \in M$ such that $u \geq w$ and $v \geq w$. Let $i \in I$ be such that $w \in M_i$. Then $u, v \in M_i$, since M_i satisfies MT1. \square

3 The graph E_P and the prime spectrum of $L_K(E_P)$

In the previous two sections we have developed all the necessary ideas to put us in position to present the key construction, in which we build a directed graph E_P from any poset P . In the sequel P will always be assumed to be nonempty.

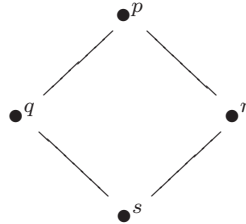
Definition 10. Given a partially ordered set (P, \leq) we define the graph E_P as follows:

$$E_P^0 = \{v_p \mid p \in P\} \quad \text{and} \quad E_P^1 = \{e_{p,q}^i \mid i \in \mathbb{N}, \text{ and } p, q \in P \text{ satisfy } p > q\},$$

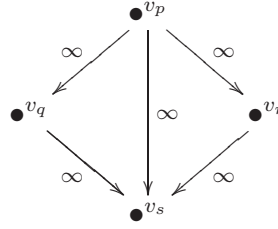
where $\mathbf{s}(e_{p,q}^i) = v_p$ and $\mathbf{r}(e_{p,q}^i) = v_q$ for all $i \in \mathbb{N}$. \square

Less formally, E_P is built from P by viewing the elements of P as vertices, and putting countably infinitely many edges from v_p to v_q whenever $p > q$ in P .

Example 11. Let P be the poset with the following Hasse diagram.



Then E_P is the graph below.



□

By inserting infinitely many edges between all connected vertices, the graph E_P and the poset $\text{Spec}(L_K(E_P))$ are endowed with particularly nice properties, as the next lemma shows. (As a reminder, for any graph F and $v \in F^0$, $\mathcal{M}(v)$ denotes the set $\{w \in F^0 \mid w \geq v\}$.)

Lemma 12. *Let (P, \leq) be a partially ordered set.*

- (1) *For all $p, q \in P$ we have $p > q$ if and only if $v_p > v_q$ if and only if $\mathcal{M}(v_p) \subset \mathcal{M}(v_q)$.*
- (2) *For all $p, q \in P$ we have $p \geq q$ if and only if $v_p \geq v_q$ if and only if $\mathcal{M}(v_p) \subseteq \mathcal{M}(v_q)$.*
- (3) *E_P is acyclic.*
- (4) *(E_P^0, \leq) is partially ordered and is order-isomorphic to (P, \leq) .*
- (5) *An ideal I of $L_K(E_P)$ is prime if and only if it is of the form $I = \langle H \rangle$, where H is a proper subset of E_P^0 such that $E_P^0 \setminus H$ satisfies MT1 and MT3.*

Proof. (1) Let $p, q \in P$ be distinct elements. If $p > q$, then $\mathbf{s}(e_{p,q}^1) = v_p$ and $\mathbf{r}(e_{p,q}^1) = v_q$, which implies that $v_p > v_q$. Conversely, if $v_p > v_q$, then there is a path $\alpha \in \text{Path}(E)$ of the form $\alpha = e_{p,a_1}^1 e_{a_1,a_2}^1 \dots e_{a_n,q}^1$ ($a_1, \dots, a_n \in P$). It follows from the definition of E_P that $p > a_1 > a_2 > \dots > a_n > q$, and hence that $p > q$, since $>$ is transitive.

Next, if $v_p > v_q$ and $w \in E_P^0$ is such that $w \geq v_p$, then clearly $w \geq v_q$. It follows that $\mathcal{M}(v_p) \subseteq \mathcal{M}(v_q)$. Moreover, it cannot be the case that $v_q \geq v_p$, since then we would have $p > q$ and $q \geq p$, which is not possible in a partially ordered set. Hence $v_q \notin \mathcal{M}(v_p)$, and therefore $\mathcal{M}(v_p) \subset \mathcal{M}(v_q)$. Conversely, if $\mathcal{M}(v_p) \subset \mathcal{M}(v_q)$, then, in particular, $v_p \in \mathcal{M}(v_q)$, and hence $v_p > v_q$, by the definition of $\mathcal{M}(v_q)$.

(2) This follows immediately from (1).

(3) By the definition of E_P , there are no loops in $\text{Path}(E_P)$. Thus for E_P to be not acyclic there must be distinct vertices $v_p, v_q \in E_P^0$ ($p, q \in P$) such that $v_p > v_q$ and $v_q > v_p$. But, by (1) this can happen only if $p > q$ and $q > p$, which is not possible, since \geq is antisymmetric. Hence E_P must be acyclic.

(4) As mentioned above, (E_P^0, \leq) is preordered. Hence, by (3), (E_P^0, \leq) is partially ordered, and by (2), $(E_P^0, \leq) \cong (P, \leq)$.

(5) Let $v_p, v_q \in E_P^0$ be two vertices ($p, q \in P$). If $v_p > v_q$, then $|\mathbf{s}^{-1}(v_p) \cap \mathbf{r}^{-1}(v_q)| = \aleph_0$, by (1) and the definition of E_P . Hence, for any subset $H \subseteq E_P^0$ either $|\mathbf{s}^{-1}(v_p) \cap \mathbf{r}^{-1}(E_P^0 \setminus H)| = 0$ or $|\mathbf{s}^{-1}(v_p) \cap \mathbf{r}^{-1}(E_P^0 \setminus H)| \geq \aleph_0$. In particular, if $H \subseteq E_P^0$ is hereditary and saturated, then $B_H = \emptyset$.

Finally, note that every subset of E_P^0 satisfies MT2 vacuously, since E_P^0 contains no regular vertices. The desired conclusion now follows from Corollary 7 and (3). □

We now relate the structure of any partially ordered set (P, \leq) to that of $\text{Spec}(L_K(E_P))$.

Proposition 13. *Let (P, \leq) be a partially ordered set and K a field. Define the function*

$$\phi : P \rightarrow \text{Spec}(L_K(E_P)) \text{ by setting } \phi(p) = \langle E_P^0 \setminus \mathcal{M}(v_p) \rangle \text{ for each } p \in P.$$

Then ϕ is an order-embedding. Moreover, ϕ is an order-isomorphism if and only if (P, \leq) satisfies DCC.

Proof. First note that for any vertex u (in any graph), $\mathcal{M}(u)$ is nonempty and satisfies MT1 and MT3. Hence, by Lemma 12(5), $\langle E_P^0 \setminus \mathcal{M}(v_p) \rangle \in \text{Spec}(L_K(E_P))$ for all $p \in P$.

Now let $p, q \in P$ and suppose that $q \leq p$. Then $\mathcal{M}(v_p) \subseteq \mathcal{M}(v_q)$, by Lemma 12(2), and hence $\langle E_P^0 \setminus \mathcal{M}(v_q) \rangle \subseteq \langle E_P^0 \setminus \mathcal{M}(v_p) \rangle$. That is, $\phi(q) \subseteq \phi(p)$. Hence ϕ is order-preserving.

Next, suppose that $\phi(q) \subseteq \phi(p)$. Then $\langle E_P^0 \setminus \mathcal{M}(v_q) \rangle \subseteq \langle E_P^0 \setminus \mathcal{M}(v_p) \rangle$. By Remark 5, we have

$$E_P^0 \setminus \mathcal{M}(v_q) = \langle E_P^0 \setminus \mathcal{M}(v_q) \rangle \cap E_P^0 \subseteq \langle E_P^0 \setminus \mathcal{M}(v_p) \rangle \cap E_P^0 = E_P^0 \setminus \mathcal{M}(v_p),$$

and hence $\mathcal{M}(v_p) \subseteq \mathcal{M}(v_q)$. By Lemma 12(2), this implies that $q \leq p$, and hence ϕ is order-reflecting, and therefore an order-embedding.

To prove the final claim, suppose that (P, \leq) satisfies DCC. To conclude that ϕ is a bijection (and hence an order-isomorphism) it suffices to show that if I is a prime ideal of $L_K(E_P)$, then $I = \langle E_P^0 \setminus \mathcal{M}(u) \rangle$ for some $u \in E_P^0$. So let I be a prime ideal of $L_K(E_P)$. By Lemma 12(5), $I = \langle H \rangle$, where H is a proper subset of E_P^0 such that $E_P^0 \setminus H$ satisfies MT1 and MT3. Let $Q = \{p \mid v_p \in E_P^0 \setminus H\} \subseteq P$. Since P satisfies DCC, so does Q , and hence Q must have a minimal element. Suppose that $p, q \in Q$ are two minimal elements. Since $v_p, v_q \in E_P^0 \setminus H$ and this set satisfies MT3, there must exist some $r \in Q$ such that $v_r \in E_P^0 \setminus H$, $v_p \geq v_r$, and $v_q \geq v_r$. By Lemma 12(2), this implies that $p \geq r$ and $q \geq r$. But since p and q are minimal, we conclude that $p = r = q$, and hence Q must have a unique minimal element r . Now since $E_P^0 \setminus H$ satisfies MT3, it follows that for any $v_p \in E_P^0 \setminus H$ we have $v_p \geq v_r$, and hence $E_P^0 \setminus H \subseteq \mathcal{M}(v_r)$. Also, if $v_p \in E^0$ is any vertex such that $v_p \geq v_r$, then $v_p \in E_P^0 \setminus H$, since this set satisfies MT1. Therefore $E_P^0 \setminus H = \mathcal{M}(v_r)$, and hence $H = E_P^0 \setminus \mathcal{M}(v_r)$, as desired.

Conversely, suppose that ϕ is an order-isomorphism. Then every prime ideal of $L_K(E_P)$ must be of the form $\langle E_P^0 \setminus \mathcal{M}(u) \rangle$ for some $u \in E_P^0$. Seeking a contradiction, suppose that (P, \leq) does not satisfy DCC. Then there is a subset $\{p_i \mid i \in \mathbb{N}\}$ of P satisfying $p_0 > p_1 > p_2 > \dots$. By Lemma 12(1),

$$\mathcal{M}(v_{p_0}) \subset \mathcal{M}(v_{p_1}) \subset \mathcal{M}(v_{p_2}) \subset \dots,$$

and by Lemma 9, $M = \bigcup_{i \in \mathbb{N}} \mathcal{M}(v_{p_i})$ is a maximal tail, since each $\mathcal{M}(v_{p_i})$ is. Hence, by Lemma 12(5), $\langle E_P^0 \setminus M \rangle$ is a prime ideal of $L_K(E_P)$. However, M cannot be of the form $M = \mathcal{M}(u)$ for any $u \in E_P^0$, since otherwise $u \in \mathcal{M}(v_{p_i})$ for some $i \in \mathbb{N}$, and then we would have $M = \mathcal{M}(v_{p_i})$. This gives the desired contradiction, and hence (P, \leq) must satisfy DCC. \square

Our next goal is to describe $\text{Spec}(L_K(E_P))$ for an arbitrary partially ordered set (P, \leq) . Towards that end we shall require the following notation.

Definition 14. Let (P, \leq) be a partially ordered set. Define a binary relation \preceq on the power set $\mathcal{P}(P)$ of P as follows. Given $S_1, S_2 \in \mathcal{P}(P) \setminus \{\emptyset\}$, write $S_1 \preceq S_2$ if for every $s_2 \in S_2$ there exists $s_1 \in S_1$ such that $s_1 \leq s_2$. If $S_1 \preceq S_2$ and $S_2 \preceq S_1$, then we write $S_1 \approx S_2$. \square

It is routine to verify that $(\mathcal{P}(P), \preceq)$ is a preordered set, that \approx is an equivalence relation on $\mathcal{P}(P)$, and that \preceq induces a partial order on the set $\mathcal{P}(P)/\approx$ of \approx -equivalence classes in $\mathcal{P}(P)$. Given $S \in \mathcal{P}(P)$ we shall denote the \approx -equivalence class of S by $[S]$.

Definition 15. Given a partially ordered set (P, \leq) , let

$$\mathcal{A}(P) = P \cup \{x_{[S]} \mid S \subseteq P \text{ is downward directed with no least element}\}.$$

Further, we extend \leq to a binary relation $\leq_{\mathcal{A}}$ on $\mathcal{A}(P)$, as follows. For all $p, q \in \mathcal{A}(P)$ let $p \leq_{\mathcal{A}} q$ if one of the following holds:

- (1) $p, q \in P$ and $p \leq q$;
- (2) $p \in P$, $q = x_{[S]} \in \mathcal{A}(P) \setminus P$, and $p \leq s$ for all $s \in S$;
- (3) $p = x_{[S]} \in \mathcal{A}(P) \setminus P$, $q \in P$, and $s \leq q$ for some $s \in S$;
- (4) $p, q \in \mathcal{A}(P) \setminus P$, $p = x_{[S]}$, $q = x_{[T]}$, and $S \preceq T$. \square

Loosely speaking, to construct $\mathcal{A}(P)$ from P we *adjoin* a greatest lower bound to each downward directed subset S of P containing no least element. (We note that S might already have a greatest lower bound in P . Even so, we still adjoint a “new” greatest lower bound for S in $\mathcal{A}(P)$.)

Example 16. Let $P = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ with the usual order. Clearly $S = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ is a subset of P which contains no least element. Clearly also any infinite subset S' of S has $[S'] = [S]$, while any finite subset of S contains a least element. So

$$\mathcal{A}(P) = P \cup \{x_{[S]}\},$$

where $0 < x_{[S]} < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$. \square

Now invoking Remark 1, we easily get the following.

Lemma 17. *Let P be any partially ordered set. Then P has DCC if and only if $\mathcal{A}(P) = P$.*

In particular, we note that the operation \mathcal{A} on posets should not be viewed as a closure operation, i.e., $\mathcal{A}(\mathcal{A}(P))$ need not equal $\mathcal{A}(P)$. (This is readily seen by considering the poset P of Example 16.)

Although we have already defined the binary relation $\leq_{\mathcal{A}}$ on $\mathcal{A}(P)$ for any poset P , we have yet to make any claims regarding relational properties of $\leq_{\mathcal{A}}$. While the reader’s likely hunch that $\leq_{\mathcal{A}}$ is a partial order on $\mathcal{A}(P)$ is indeed true, a direct first-principles verification of this fact requires some significant effort (including checking numerous cases). As it turns out, this fact will follow as a nice byproduct of the following theorem, which is the main result of this article.

Theorem 18. *Let (P, \leq) be a partially ordered set and K a field. Then $(\mathcal{A}(P), \leq_{\mathcal{A}})$ is a partially ordered set, and*

$$(\text{Spec}(L_K(E_P)), \subseteq) \cong (\mathcal{A}(P), \leq_{\mathcal{A}}).$$

Proof. Let $\phi : P \rightarrow \text{Spec}(L_K(E_P))$ be as in Proposition 13. The goal of Claims 1 and 2 below is to extend ϕ to a function $\varphi : \mathcal{A}(P) \rightarrow \text{Spec}(L_K(E_P))$. Then in Claims 3 and 4 we establish that φ is a bijection that respects the binary relations $\leq_{\mathcal{A}}$ and \subseteq . Both assertions of the theorem will then follow.

Claim 1: If $I \in \text{Spec}(L_K(E_P)) \setminus \phi(P)$, then there exists a downward directed subset $S \subseteq P$ with no least element such that $I = \bigcap_{s \in S} \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle$.

To see this, we note that, by Lemma 12(5), any such ideal I must be of the form $I = \langle H \rangle$, where $H \subset E_P^0$ is such that $M = E_P^0 \setminus H$ satisfies MT1 and MT3. Let $S = \{s \mid v_s \in M\} \subseteq P$. Then $M = \bigcup_{s \in S} \mathcal{M}(v_s)$, since M satisfies MT1. By Lemma 12(4), (M, \leq) is a partially ordered set (since $M \subseteq E_P^0$) and $(M, \leq) \cong (S, \leq)$. Since M satisfies MT3, (M, \leq) is downward directed, and hence so is (S, \leq) . Moreover, S cannot have a least element $s \in S$, since otherwise we would have $M = \mathcal{M}(v_s)$, and therefore $I = \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle = \phi(v_s)$, contrary to the choice of I . Now,

$$I = \langle E_P^0 \setminus M \rangle = \langle E_P^0 \setminus \bigcup_{s \in S} \mathcal{M}(v_s) \rangle = \langle \bigcap_{s \in S} E_P^0 \setminus \mathcal{M}(v_s) \rangle,$$

and hence we seek to show that the last expression in the previous display is equal to $\bigcap_{s \in S} \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle$.

Let $x \in \langle \bigcap_{s \in S} E_P^0 \setminus \mathcal{M}(v_s) \rangle$ be any element. Then $x \in \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle$ for all $s \in S$, and hence $x \in \bigcap_{s \in S} \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle$, showing that

$$\langle \bigcap_{s \in S} E_P^0 \setminus \mathcal{M}(v_s) \rangle \subseteq \bigcap_{s \in S} \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle.$$

For the opposite inclusion, note that since S is downward directed, $\{\langle E_P^0 \setminus \mathcal{M}(v_s) \rangle \mid s \in S\}$ is a downward directed subset of $\text{Spec}(L_K(E))$, by Lemma 12(2), and hence $\bigcap_{s \in S} \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle$ is a prime ideal, by Proposition 3. It follows, by Lemma 12(5), that $\bigcap_{s \in S} \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle = \langle H' \rangle$ for some $H' \subset E_P^0$. Let $v \in H'$ be any vertex. Then for all $s \in S$ we have $v \in \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle$, and hence $v \in E_P^0 \setminus \mathcal{M}(v_s)$, by Remark 5. Thus $v \in \bigcap_{s \in S} E_P^0 \setminus \mathcal{M}(v_s)$, from which we conclude that

$$\bigcap_{s \in S} \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle = \langle H' \rangle \subseteq \langle \bigcap_{s \in S} E_P^0 \setminus \mathcal{M}(v_s) \rangle,$$

and therefore

$$\bigcap_{s \in S} \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle = \langle \bigcap_{s \in S} E_P^0 \setminus \mathcal{M}(v_s) \rangle.$$

This establishes Claim 1.

For any downward directed subset $S \subseteq P$ with no least element, define

$$I_{[S]} = \bigcap_{s \in S} \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle.$$

Claim 2: If $S, S' \subseteq P$ are downward directed subsets with no least elements, such that $S \approx S'$, then $I_{[S]} = I_{[S']}$.

Noting that if $S \approx S'$, then $\bigcup_{s \in S} \mathcal{M}(v_s) = \bigcup_{s \in S'} \mathcal{M}(v_s)$, by Lemma 12(2), this claim follows by using the final displayed conclusion of Claim 1, which yields

$$I_{[S]} = \bigcap_{s \in S} \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle = \langle \bigcap_{s \in S} E_P^0 \setminus \mathcal{M}(v_s) \rangle = \langle \bigcap_{s \in S'} E_P^0 \setminus \mathcal{M}(v_s) \rangle = \bigcap_{s \in S'} \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle = I_{[S']}.$$

We now define the function $\varphi : \mathcal{A}(P) \rightarrow \text{Spec}(L_K(E_P))$ by setting

$$\varphi(p) = \begin{cases} \phi(p) & \text{if } p \in P \\ I_{[S]} & \text{if } p = x_{[S]} \in \mathcal{A}(P) \setminus P \end{cases}.$$

By Claim 2, φ is well-defined, and by Claim 1, φ is surjective.

Claim 3: φ is a bijection.

To establish this claim, we recall that by Proposition 13, $\phi : P \rightarrow \phi(P)$ is a bijection, and hence it is enough to show that if $S, S' \subseteq P$ are downward directed subsets with no least elements such that $S \not\approx S'$, then $I_{[S]} \neq I_{[S']}$. For this purpose suppose, without loss of generality, that $S \not\preceq S'$, that is there exists $s' \in S'$ such that $s \not\preceq s'$ for all $s \in S$. Then by Lemma 12(2), $\mathcal{M}(v_{s'}) \not\subseteq \mathcal{M}(v_s)$ for all $s \in S$, and hence $v_{s'} \notin \mathcal{M}(v_s)$ for all $s \in S$. It follows that $\bigcup_{s \in S'} \mathcal{M}(v_s) \not\subseteq \bigcup_{s \in S} \mathcal{M}(v_s)$, and therefore, by Remark 5,

$$I_{[S]} = \langle \bigcap_{s \in S} E_P^0 \setminus \mathcal{M}(v_s) \rangle \neq \langle \bigcap_{s \in S'} E_P^0 \setminus \mathcal{M}(v_s) \rangle = I_{[S]},$$

as desired.

Claim 4: $p \leq_{\mathcal{A}} q$ if and only if $\varphi(p) \subseteq \varphi(q)$ for any pair $p, q \in \mathcal{A}(P)$.

With Definition 15 in mind, there are four cases to check. First, by Proposition 13, ϕ is an order-embedding, so whenever $p, q \in P$ we have $p \leq_{\mathcal{A}} q$ if and only if $p \leq q$ if and only if $\varphi(p) \subseteq \varphi(q)$. Second, suppose that $p \in P$ and $q = x_{[S]} \in \mathcal{A}(P) \setminus P$. Then

$$\begin{aligned} p \leq_{\mathcal{A}} q &\Leftrightarrow p \leq s \text{ for all } s \in S \text{ (by Definition 15)} \\ &\Leftrightarrow \phi(p) \subseteq \phi(s) \text{ for all } s \in S \text{ (by Proposition 13)} \\ &\Leftrightarrow \langle E_P^0 \setminus \mathcal{M}(v_p) \rangle \subseteq \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle \text{ for all } s \in S \\ &\Leftrightarrow \langle E_P^0 \setminus \mathcal{M}(v_p) \rangle \subseteq I_{[S]} \\ &\Leftrightarrow \varphi(p) \subseteq \varphi(q). \end{aligned}$$

Third, suppose that $p = x_{[S]} \in \mathcal{A}(P) \setminus P$ and $q \in P$. Note that if

$$I_{[S]} = \langle \bigcap_{s \in S} E_P^0 \setminus \mathcal{M}(v_s) \rangle \subseteq \langle E_P^0 \setminus \mathcal{M}(v_q) \rangle,$$

then $\bigcap_{s \in S} E_P^0 \setminus \mathcal{M}(v_s) \subseteq E_P^0 \setminus \mathcal{M}(v_q)$, by Remark 5, and hence $\bigcup_{s \in S} \mathcal{M}(v_s) \supseteq \mathcal{M}(v_q)$. Thus $v_q \in \mathcal{M}(v_s)$ for some $s \in S$, and therefore $\mathcal{M}(v_q) \subseteq \mathcal{M}(v_s)$. It follows that $I_{[S]} \subseteq$

$\langle E_P^0 \setminus \mathcal{M}(v_q) \rangle$ if and only if $\langle E_P^0 \setminus \mathcal{M}(v_s) \rangle \subseteq \langle E_P^0 \setminus \mathcal{M}(v_q) \rangle$ for some $s \in S$. Therefore

$$\begin{aligned}
p \leq_{\mathcal{A}} q &\Leftrightarrow s \leq q \text{ for some } s \in S \text{ (by Definition 15)} \\
&\Leftrightarrow \phi(s) \subseteq \phi(q) \text{ for some } s \in S \text{ (by Proposition 13)} \\
&\Leftrightarrow \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle \subseteq \langle E_P^0 \setminus \mathcal{M}(v_q) \rangle \text{ for some } s \in S \\
&\Leftrightarrow I_{[S]} \subseteq \langle E_P^0 \setminus \mathcal{M}(v_q) \rangle \\
&\Leftrightarrow \varphi(p) \subseteq \varphi(q).
\end{aligned}$$

Fourth and finally, suppose that $p, q \in \mathcal{A}(P) \setminus P$, $p = x_{[S]}$, and $q = x_{[T]}$. Then

$$\begin{aligned}
p \leq_{\mathcal{A}} q &\Leftrightarrow \text{for all } t \in T \text{ there exists } s \in S \text{ such that } s \leq t \text{ (by Definition 15)} \\
&\Leftrightarrow \text{for all } t \in T \text{ there exists } s \in S \text{ such that } \phi(s) \subseteq \phi(t) \text{ (by Proposition 13)} \\
&\Leftrightarrow \text{for all } t \in T \text{ there exists } s \in S \text{ such that } \langle E_P^0 \setminus \mathcal{M}(v_s) \rangle \subseteq \langle E_P^0 \setminus \mathcal{M}(v_t) \rangle \\
&\Leftrightarrow I_{[S]} \subseteq \langle E_P^0 \setminus \mathcal{M}(v_t) \rangle \text{ for all } t \in T \\
&\Leftrightarrow I_{[S]} \subseteq I_{[T]} \\
&\Leftrightarrow \varphi(p) \subseteq \varphi(q).
\end{aligned}$$

Thus $p \leq_{\mathcal{A}} q$ if and only if $\varphi(p) \subseteq \varphi(q)$ for all $p, q \in \mathcal{A}(P)$, which establishes Claim 4.

From Claims 3 and 4 we conclude that $(\mathcal{A}(P), \leq_{\mathcal{A}})$ is a partially ordered set, since $(\text{Spec}(L_K(E_P)), \subseteq)$ is, and that $(\text{Spec}(L_K(E_P)), \subseteq) \cong (\mathcal{A}(P), \leq_{\mathcal{A}})$. \square

Using the fact that $(\mathcal{A}(P), \leq_{\mathcal{A}})$ contains a copy of (P, \leq) , or by invoking Proposition 13, we get the following.

Corollary 19. *For any partially ordered set (P, \leq) there is an order-embedding of (P, \leq) into $(\text{Spec}(L_K(E_P)), \subseteq)$.*

We close this section by noting that, by Lemma 17, Theorem 18 is the generalization of the final statement of Proposition 13 to all partially ordered sets.

4 The posets $\mathcal{A}(P)$ and $\mathcal{R}(P)$

In this section we identify, up to poset isomorphism, those partially ordered sets which can be expressed in the form $(\mathcal{A}(P), \leq_{\mathcal{A}})$.

Definition 20. Let (P, \leq) be a partially ordered set. For any $S \subseteq P$ and $p \in P$, we write $p = \text{glb}(S)$ in case S has a greatest lower bound in P , and it is equal to p . Also, let

$$\mathcal{R}(P) = \{p \in P \mid p \neq \text{glb}(S) \text{ for all } S \subseteq P \text{ downward directed without least element}\}.$$

We view $\mathcal{R}(P)$ as a partially ordered subset of P . \square

Rephrased, $\mathcal{R}(P)$ is constructed from P by *removing* each element of P which arises as the greatest lower bound of a downward directed subset of P not containing a least element. As we shall see in Proposition 25, the operation \mathcal{R} on posets “undoes” the effects of applying \mathcal{A} . Understanding precisely how these two operations interact with each other will be key to classifying the posets of the $(\mathcal{A}(P), \leq_{\mathcal{A}})$.

Example 21. Let P be the poset $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ of Example 16. Then $\mathcal{R}(P) = S = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$. This is because $0 = \text{glb}(S)$ and S has no least element, so $0 \notin \mathcal{R}(P)$. On the other hand, each element $\frac{1}{n}$ is in $\mathcal{R}(P)$, because if $\frac{1}{n}$ is the greatest lower bound for some subset Q of P , then Q must be finite, and hence $\frac{1}{n} \in Q$. This also shows that $\mathcal{R}(S) = S$.

Observing that $P \cong \mathcal{A}(S)$, the above computation implies that $\mathcal{R}(\mathcal{A}(S)) = S$. Arguing in a similar manner, we see also that $\mathcal{R}(\mathcal{A}(P)) = P$. (In this case $x_{[S]} \notin \mathcal{R}(\mathcal{A}(P))$ because $x_{[S]} = \text{glb}(S)$, while $0 \in \mathcal{R}(\mathcal{A}(P))$ because 0 is *not* the greatest lower bound of S in $\mathcal{A}(P)$.)

Now applying the two operations in the opposite order, we have $\mathcal{A}(\mathcal{R}(S)) = \mathcal{A}(S) \cong P \not\cong S$, whereas $\mathcal{A}(\mathcal{R}(P)) = \mathcal{A}(S) \cong P$. \square

We require a bit more notation and several preliminary results in order to classify the partially ordered sets that can be expressed in the form $(\mathcal{A}(P), \leq_{\mathcal{A}})$.

Notation 22. We assign the indicated names in case the poset (P, \leq) satisfies the following germane properties.

- (GLB) Every downward directed subset of P has a greatest lower bound in P .
- (DC) For every downward directed subset $S \subseteq P$ and every $p \in \mathcal{R}(P)$ satisfying $p \geq \text{glb}(S)$, we have $p \geq s$ for some $s \in S$.
- (DD) For every downward directed subset $S \subseteq P$ there exists a downward directed subset $T \subseteq \mathcal{R}(P)$ such that $\text{glb}(S) = \text{glb}(T)$.

Of course GLB stands for *greatest lower bound*. We have chosen the name DC to stand for *directed compatibility*, while DD connotes *directed discreteness*. \square

Our reason for describing the condition in DD as “discreteness” will be clarified in Proposition 38.

Lemma 23. Let (P, \leq) be a partially ordered set, let $S \subseteq P$ be a downward directed subset such that $\text{glb}(S) \in P$, and let

$$S' = S \cup \{t \in P \mid \exists s \in S \text{ such that } s \leq t\}.$$

Then the following hold.

- (1) The set S' is downward directed, and $\text{glb}(S) = \text{glb}(S')$.
- (2) If S has no least element, then neither does S' , and for all $s \in S$ there exists $t \in S$ such that $t < s$.

Proof. (1) Let $t_1, t_2 \in S'$ be any elements, and let $s_1, s_2 \in S$ be such that $s_1 \leq t_1$ and $s_2 \leq t_2$. Since S is downward directed, there exists $p \in S \subseteq S'$ such that $p \leq s_1$ and $p \leq s_2$. Thus $p \leq t_1$ and $p \leq t_2$, showing that S' is downward directed.

Since for all $t \in S'$ there exists $s \in S$ with $s \leq t$, it follows that $\text{glb}(S) \leq t$. Thus $\text{glb}(S)$ is a lower bound for S' . On the other hand, since $S \subseteq S'$, any lower bound for S' must be $\leq \text{glb}(S)$. Hence $\text{glb}(S) = \text{glb}(S')$.

(2) If S' were to have a least element s , then $s \in S$, by the definition of S' , and $s = \text{glb}(S') = \text{glb}(S)$ would also be the least element of S .

Now, let $s \in S$ be any element. Assuming that S has no least element, there exists $p \in S$ such that $s \not\leq p$. Since S is downward directed, there exists $t \in S$ such that $t \leq s$ and $t \leq p$. Finally, $s \not\leq p$ implies that $t \neq s$, and hence $t < s$. \square

The poset S' constructed in the previous lemma happens to be a *filter* in P , i.e., a downward directed subset with the property that for all $s \in S'$ and $t \in P$, $s \leq t$ implies that $t \in S'$. In fact, S' is the filter of P generated by S , i.e., the smallest filter containing S .

Lemma 24. *Let (P, \leq) be a partially ordered set, and let $S \subseteq \mathcal{A}(P)$ be a downward directed subset with no least element. Then there exists $S'' \subseteq P$ downward directed with no least element, such that $\text{glb}(S) = \text{glb}(S'')$ and for all $s'' \in S''$ there exists $s \in S$ satisfying $s \leq_A s''$.*

Proof. We observe that $\text{glb}(S) \in \mathcal{A}(P)$, since $(\mathcal{A}(P), \leq_A)$ satisfies GLB, by Theorem 18 and Proposition 3.

We begin by setting

$$S' = S \cup \{t \in \mathcal{A}(P) \mid \exists s \in S \text{ such that } s \leq_A t\}.$$

Then, by Lemma 23, S' is downward directed, $\text{glb}(S) = \text{glb}(S')$, and S' has no least element. Now let $S'' = S' \cap P$. To see that S'' is also downward directed, let $s, t \in S''$. Then there exists $r \in S'$ such that $r \leq_A s$ and $r \leq_A t$, since S' is downward directed. If $r \in P$, then $r \in S''$, as required. We therefore assume that $r \in \mathcal{A}(P) \setminus P$, and hence that $r = x_{[T]}$ for some downward directed subset $T \subseteq P$ with no least element. Since $x_{[T]} \leq_A s$ and $x_{[T]} \leq_A t$, there exist $p, q \in T$ such that $p \leq s$ and $q \leq t$, by Definition 15(3). Since T is downward directed, there exists $u \in T \subseteq P$ such that $u \leq p \leq s$ and $u \leq q \leq t$. By the definition of S' , since $r \in S'$, there exists $v \in S$ such that $v \leq_A r$. Then $u \in S'$, since $v \leq_A r = x_{[T]} \leq_A u$, and hence $u \in S''$, showing that S'' is downward directed.

Note that $\text{glb}(S') \leq_A \text{glb}(S'')$, since $S'' \subseteq S'$. Also, for all $x_{[T]} \in S' \setminus S''$ we have $\text{glb}(S'') \leq_A x_{[T]}$, since $x_{[T]} = \text{glb}(T)$ and $T \subseteq S''$. Hence $\text{glb}(S'') \leq_A \text{glb}(S')$, and therefore $\text{glb}(S) = \text{glb}(S') = \text{glb}(S'')$. Also, S'' has no least element since if there were such an element $s \in S''$, then we would have $s = \text{glb}(S'') = \text{glb}(S')$, implying that S' has a least element. Finally, for all $s'' \in S''$ there exists $s \in S$ satisfying $s \leq_A s''$, since $S'' \subseteq S'$. \square

Proposition 25. *Let P be any partially ordered set. Then $\mathcal{R}(\mathcal{A}(P)) = P$.*

Proof. Let $S \subseteq \mathcal{A}(P)$ be a downward directed subset with no least element. Then, by Lemma 24, there is a downward directed subset $S'' \subseteq P$ with no least element, such that $\text{glb}(S) = \text{glb}(S'')$. Since $S'' \subseteq P$, we have $\text{glb}(S'') = x_{[S'']}$, and hence $\text{glb}(S'') = \text{glb}(S) \notin P$. As $S \subseteq \mathcal{A}(P)$ was arbitrary, this implies that $P \subseteq \mathcal{R}(\mathcal{A}(P))$. But the reverse inclusion $\mathcal{R}(\mathcal{A}(P)) \subseteq P$ holds as well, since every element of $\mathcal{A}(P) \setminus P$ is the greatest lower bound of a downward directed subset with no least element, by Definition 15. Thus $\mathcal{R}(\mathcal{A}(P)) = P$. \square

Lemma 26. *Let (P, \leq) be a partially ordered set. Then $(\mathcal{A}(P), \leq_A)$ satisfies GLB, DC, and DD.*

Proof. For any ring R , $(\text{Spec}(R), \subseteq)$ satisfies GLB, by Proposition 3. Hence, $(\mathcal{A}(P), \leq_A)$ satisfies GLB, by Theorem 18.

To show that $(\mathcal{A}(P), \leq_A)$ satisfies DD, let $S \subseteq \mathcal{A}(P)$ be a downward directed subset. We wish to find a downward directed subset $T \subseteq \mathcal{R}(\mathcal{A}(P))$ such that $\text{glb}(S) = \text{glb}(T)$. By Lemma 24 and Proposition 25, we may assume that S has a least element s . If $s \in P$, then

we may take $T = \{s\}$, which is in $\mathcal{R}(\mathcal{A}(P))$, by Proposition 25. Otherwise $s = x_{[T]}$ for some downward directed subset $T \subseteq P = \mathcal{R}(\mathcal{A}(P))$ with no least element, by Definition 15, and hence $\text{glb}(S) = x_{[T]} = \text{glb}(T)$.

Finally, to show that $(\mathcal{A}(P), \leq_{\mathcal{A}})$ satisfies DC, let $S \subseteq \mathcal{A}(P)$ be downward directed, and let $p \in \mathcal{R}(\mathcal{A}(P)) = P$ be such that $\text{glb}(S) \leq_{\mathcal{A}} p$. We wish to find $s \in S$ such that $s \leq_{\mathcal{A}} p$. If S has a least element s , then $s = \text{glb}(S)$, from which the desired conclusion follows. Therefore, assume that S has no least element. Then $\text{glb}(S) \in \mathcal{A}(P) \setminus P$, since $\mathcal{R}(\mathcal{A}(P)) = P$ (again by Proposition 25), and hence $\text{glb}(S) = x_{[S']}$ for some downward directed subset $S' \subseteq P$ with no least element. Moreover, by Lemma 24, we may assume that for every $s' \in S'$ there exists $s \in S$ such that $s \leq_{\mathcal{A}} s'$. Thus $s' \leq p$ for some $s' \in S'$, by Definition 15(3), and hence $s \leq_{\mathcal{A}} p$ for some $s \in S$. \square

We now present a description of those partially ordered sets which arise as $\mathcal{A}(Q)$ for some partially ordered set Q . The description will be given from three points of view: in terms of the poset operations \mathcal{A} and \mathcal{R} , in terms of germane poset properties, and in terms of prime spectra of Leavitt path algebras.

Theorem 27. *The following are equivalent for any partially ordered set (P, \leq) .*

- (1) $(P, \leq) \cong (\mathcal{A}(\mathcal{R}(P)), \leq_{\mathcal{A}})$.
- (2) $(P, \leq) \cong (\mathcal{A}(P'), \leq'_{\mathcal{A}})$ for some partially ordered set (P', \leq') .
- (3) $(P, \leq) \cong (\text{Spec}(L_K(E_{P'})), \subseteq)$ for any field K and some partially ordered set (P', \leq') .
- (4) $(\mathcal{R}(\mathcal{A}(P)), \leq) \cong (\mathcal{A}(\mathcal{R}(P)), \leq_{\mathcal{A}})$.
- (5) (P, \leq) satisfies GLB, DC, and DD.

Proof. (1) implies (2) tautologically, (2) and (3) are equivalent by Theorem 18, (1) and (4) are equivalent by Proposition 25, and (2) implies (5) by Lemma 26. So it suffices to assume (5), and show that (1) holds.

We define the map $\psi : P \rightarrow \mathcal{A}(\mathcal{R}(P))$ by setting

$$\psi(p) = \begin{cases} p & \text{if } p \in \mathcal{R}(P) \\ x_{[S]} & \text{if } p = \text{glb}(S) \text{ for } S \subseteq \mathcal{R}(P) \text{ downward directed without least element} \end{cases}$$

To show that ψ is well-defined, note that if $p \in P \setminus \mathcal{R}(P)$, then $p = \text{glb}(S)$ for some downward directed $S \subseteq P$ with no least element. Since (P, \leq) satisfies DD, S may be chosen such that $S \subseteq \mathcal{R}(P)$. Also since (P, \leq) satisfies DC, it is easy to see that if $T \subseteq \mathcal{R}(P)$ is another downward directed subset such that $p = \text{glb}(T)$, then $[T] = [S]$. It follows, by Definition 15, that ψ is well-defined and injective. From the definition of $\mathcal{A}(\mathcal{R}(P))$ and the fact that (P, \leq) satisfies GLB it is also evident that ψ is surjective. Thus it remains to show that ψ is an order-embedding. As in the proof of Claim 4 of Theorem 18, there are four cases to check.

Let $p, q \in P$. First, if $p, q \in \mathcal{R}(P)$, then $p = \psi(p)$ and $q = \psi(q)$, and hence $p \leq q$ if and only if $\psi(p) \leq_{\mathcal{A}} \psi(q)$. Second, if $p \in \mathcal{R}(P)$ and $q \in P \setminus \mathcal{R}(P)$, then, as above, $q = \text{glb}(S)$ for

some downward directed $S \subseteq \mathcal{R}(P)$ with no least element. Thus

$$\begin{aligned} p \leq q &\Leftrightarrow p \leq s \text{ for all } s \in S \\ &\Leftrightarrow p \leq_{\mathcal{A}} x_{[S]} \text{ (by Definition 15)} \\ &\Leftrightarrow \psi(p) \leq_{\mathcal{A}} \psi(q). \end{aligned}$$

Third, suppose that $p \in P \setminus \mathcal{R}(P)$ and $q \in \mathcal{R}(P)$. Again, $p = \text{glb}(S)$ for some downward directed $S \subseteq \mathcal{R}(P)$ with no least element. Then

$$\begin{aligned} p \leq q &\Leftrightarrow s \leq q \text{ for some } s \in S \text{ (since } (P, \leq) \text{ satisfies DC)} \\ &\Leftrightarrow x_{[S]} \leq_{\mathcal{A}} q \text{ (by Definition 15)} \\ &\Leftrightarrow \psi(p) \leq_{\mathcal{A}} \psi(q). \end{aligned}$$

Fourth and finally, suppose that $p, q \in P \setminus \mathcal{R}(P)$, and write $p = \text{glb}(S)$, $q = \text{glb}(T)$ for some downward directed $S, T \subseteq \mathcal{R}(P)$ with no least element. Then

$$\begin{aligned} p \leq q &\Leftrightarrow \text{for all } t \in T \text{ there exists } s \in S \text{ such that } s \leq t \text{ (since } (P, \leq) \text{ satisfies DC)} \\ &\Leftrightarrow x_{[S]} \leq_{\mathcal{A}} x_{[T]} \text{ (by Definition 15)} \\ &\Leftrightarrow \psi(p) \leq_{\mathcal{A}} \psi(q). \end{aligned}$$

Hence $\psi : P \rightarrow \mathcal{A}(\mathcal{R}(P))$ is an order-embedding, as desired. \square

In particular, we may recast the implication (5) \Rightarrow (3) of Theorem 27 as a partial answer to the aforementioned Realization Question.

Corollary 28. *Let (P, \leq) be a partially ordered set which satisfies conditions GLB, DC, and DD, and let K be any field. Then there exists a K -algebra A for which $(\text{Spec}(A), \subseteq) \cong (P, \leq)$.*

Corollary 28 is fairly far-reaching, but not complete. First, we know by Proposition 3 that condition GLB is a necessary condition in any well-posed Realization Question. Second, it is an open question as to whether condition DD is necessary in general. (Further discussion is presented in Section 5.) Third, a reconsideration of the graph E of Example 8 above yields a Leavitt path algebra $L_K(E)$ for which $\text{Spec}(L_K(E))$ does not satisfy property DC. Specifically, we see that the descending chains $S_M = \{\langle E^0 \setminus M_i \rangle \mid i \in \mathbb{Z}^+\}$ and $S_N = \{\langle E^0 \setminus N_i \rangle \mid i \in \mathbb{Z}^+\}$ in $\text{Spec}(L_K(E))$ have $\text{glb}(S_M) = 0 = \text{glb}(S_N)$, and that each element of $S_M \cup S_N$ is in $\mathcal{R}(\text{Spec}(L_K(E)))$, but no element of S_M is comparable to any element of S_N .

We conclude this section with a few observations on how to recognize those elements of a partially ordered set (P, \leq) which belong to $\mathcal{R}(P)$, since these feature prominently in the results above. Our focus is on the case where (P, \leq) is the prime spectrum of a Leavitt path algebra.

Lemma 29. *The following are equivalent for any partially ordered set (P, \leq) and $p \in P$.*

- (1) $p \notin \mathcal{R}(P)$.
- (2) *There exists a downward directed $S \subseteq P$ such that $p = \text{glb}(S)$ but $p \notin S$.*

Proof. (1) \Rightarrow (2) If $p \notin \mathcal{R}(P)$, then there exists a downward directed subset $S \subseteq P$ with no least element such that $p = \text{glb}(S)$. Then $p \notin S$, since otherwise S would have a least element.

(2) \Rightarrow (1) Let $S \subseteq P$ be as in (2). If S were to have a least element p' , then $p' \leq \text{glb}(S) = p$, and hence $p' = p$. But this would contradict $p \notin S$, and hence S cannot have a least element. \square

Lemma 30. *Let R be a ring and $I \in \text{Spec}(R) \setminus \mathcal{R}(\text{Spec}(R))$. Then*

$$I = \bigcap \{Q \in \text{Spec}(R) \mid I \subsetneq Q\}.$$

Proof. Let $T = \{Q \in \text{Spec}(R) \mid I \subsetneq Q\}$. By Lemma 29, there is a downward directed $S \subseteq \text{Spec}(R)$ such that $I = \text{glb}(S) = \bigcap_{Q \in S} Q$ but $I \notin S$. Then $S \subseteq T$, and hence $\bigcap_{Q \in T} Q \subseteq \bigcap_{Q \in S} Q$. But $I \subseteq \bigcap_{Q \in T} Q$, and therefore $I = \bigcap_{Q \in T} Q$. \square

A prime ideal I that is the intersection of all the prime ideals strictly containing it (as in the previous lemma) is called *not locally closed*. A classification of the locally closed prime ideals in $L_K(E)$, for E finite, is given in [2].

We note that the converse of Lemma 30 is not true. For instance, if $R = \mathbb{Z}$, then the prime ideal $\{0\}$ is the intersection over all the nonzero prime ideals of \mathbb{Z} , but $\{0\} \in \mathcal{R}(\text{Spec}(\mathbb{Z}))$, since there are clearly no downward directed sets in $\text{Spec}(\mathbb{Z})$ which lack a least element.

Proposition 31. *Let R be a ring, $Q \subseteq R$ an ideal, and $e \in R \setminus Q$ an idempotent. Then there exists an ideal P_e of R maximal with respect to not containing e , such that $Q \subseteq P_e$. Moreover, $P_e \in \mathcal{R}(\text{Spec}(R))$.*

Proof. Let S be the set of all the ideals of R that contain Q but not e . Then it is easy to see that the union of any chain of ideals from S is an ideal that contains Q but not e . Hence, by Zorn's Lemma, there exists an ideal P_e of R maximal with respect to not containing e , such that $Q \subseteq P_e$.

To show that P_e is prime, suppose that I and J are ideals such that $I, J \not\subseteq P_e$. Then $P_e \subsetneq P_e + I$ and $P_e \subsetneq P_e + J$, which implies that $e \in P_e + I$ and $e \in P_e + J$, by the maximality of P_e . Hence

$$e = e^2 \in (P_e + I)(P_e + J) \subseteq P_e + IJ,$$

and therefore $P_e \subsetneq P_e + IJ$. It follows that $IJ \not\subseteq P_e$, and thus P_e is prime.

Finally, let $T = \{I \in \text{Spec}(R) \mid P_e \subsetneq I\}$. Then $e \in I$ for all $I \in T$, by the maximality of P_e . Hence $e \in \bigcap_{I \in T} I$, and therefore $P_e \subsetneq \bigcap_{I \in T} I$. Thus, by Lemma 30, we conclude that $P_e \in \mathcal{R}(\text{Spec}(R))$. \square

Proposition 32. *Let K be a field, E a graph, $P \in \text{Spec}(L_K(E))$, and $H = P \cap E^0$. If $E^0 \setminus H = \mathcal{M}(u)$ for some $u \in E^0$, then $P \in \mathcal{R}(\text{Spec}(L_K(E)))$. In this situation either u is not regular or it is the source of a cycle.*

Proof. Seeking a contradiction, suppose that $E^0 \setminus H = \mathcal{M}(u)$ but $P \notin \mathcal{R}(\text{Spec}(L_K(E)))$. Then there exists a downward directed $S \subseteq \text{Spec}(R)$ such that $P = \bigcap_{Q \in S} Q$ but $P \notin S$, by Lemma 29. For each $Q \in S$ let $H_Q = Q \cap E^0$. Then $H = \bigcap_{Q \in S} H_Q$, and hence $\mathcal{M}(u) = \bigcup_{Q \in S} (E^0 \setminus H_Q)$. Thus, $u \in E^0 \setminus H_Q$ for some $Q \in S$. Since $E^0 \setminus H_Q$ satisfies MT1

(as noted in Section 1), we have $\mathcal{M}(u) \subseteq E^0 \setminus H_Q$. Thus $\mathcal{M}(u) = E^0 \setminus H_Q$, and therefore $H = H_Q$.

Since $P \subsetneq Q$, by Theorem 4, $H = H_Q$ is possible only in the following three situations.

- (1) $P = \langle H \cup \{v^H \mid v \in B_H\} \rangle$ and $Q = \langle H \cup \{v^H \mid v \in B_H\} \cup \{f(c)\} \rangle$, where $c \in \text{Path}(E) \setminus E^0$ is a WK cycle having source $u \in E^0$, and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.
- (2) $P = \langle H \cup \{v^H \mid v \in B_H \setminus \{u\}\} \rangle$ and $Q = \langle H \cup \{v^H \mid v \in B_H\} \rangle$.
- (3) $P = \langle H \cup \{v^H \mid v \in B_H \setminus \{u\}\} \rangle$ and $Q = \langle H \cup \{v^H \mid v \in B_H\} \cup \{f(c)\} \rangle$, where $c \in \text{Path}(E) \setminus E^0$ is a WK cycle having source $u \in E^0$, and $f(x)$ is an irreducible polynomial in $K[x, x^{-1}]$.

By [10, Section 5], which describes the prime ideals of $L_K(E)$ having the same intersection with E^0 , in cases (1) and (2) there are no prime ideals strictly between P and Q , while in case (3) the only prime ideal strictly between P and Q is $\langle H \cup \{v^H \mid v \in B_H\} \rangle$. But, by Lemma 23(2), for every $Q \in S$ there must exist $Q' \in S$ such that $Q' \subsetneq Q$ (where necessarily $P \subsetneq Q'$), giving the desired contradiction.

As noted in Remark 5, since P is an ideal, H must be saturated. Hence $\mathcal{M}(u)$ satisfies MT2, as noted in Section 1. It is easy to see that this can happen only if either u is not regular or it is the source of a cycle □

5 Condition KAP

We start this concluding section by revisiting a question mentioned in Section 2 (following Theorem 2), one of the central currently-unresolved questions regarding the prime spectra of arbitrary rings.

Definition 33. We say that a partially ordered set (P, \leq) satisfies (KAP) if the following condition holds: for all $p, q \in P$ such that $p < q$, there exist $p', q' \in P$ such that $p \leq p' < q' \leq q$, and no $t \in P$ satisfies $p' < t < q'$. □

Question 34. *Is there a ring R such that $(\text{Spec}(R), \subseteq)$ does not satisfy KAP?*

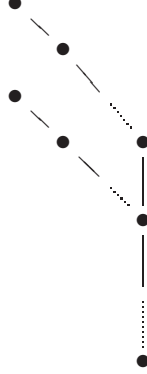
With the comments made subsequent to Corollary 28 as context, we ask the following.

Question 35. *Is there a ring R such that $(\text{Spec}(R), \subseteq)$ does not satisfy DD? (Moreover, if so, can such an R be a Leavitt path algebra?)*

Although on first glance the two conditions KAP and DD seem quite different, it turns out that there is a connection between them, one which relates Questions 34 and 35 to each other. We establish this connection in Proposition 38.

In general, conditions KAP and DD are independent of one another, as the next two examples show.

Example 36. For each $i \in \mathbb{Z}^-$, let Z_i be a copy of \mathbb{Z}^- , let Z_0 be a copy of \mathbb{Z}^- with a least element $-\infty$ adjoined, and let $P = \bigsqcup_{i \in \mathbb{Z}^- \cup \{0\}} Z_i$. Define a relation \leq_P on all elements $p, q \in P$ by $p \leq_P q$ if and only if either $p, q \in Z_i$ for some $i \in \mathbb{Z}^- \cup \{0\}$ and $p \leq q$, or $p \in Z_0 \setminus \{-\infty\}$ and $q \in Z_p$. Pictorially, P can be represented as follows.



Then it is easy to see that (P, \leq_P) is a partially ordered set. Moreover, since every element of $P \setminus \{-\infty\}$ has an immediate successor, P satisfies KAP. Also, $\mathcal{R}(P) = P \setminus Z_0$, from which it follows that every downward directed subset of $\mathcal{R}(P)$ is contained in some Z_i ($i < 0$). Thus no downward directed subset of $\mathcal{R}(P)$ has $-\infty$ as the greatest lower bound, and hence P does not satisfy DD. \square

Example 37. Let $[0, 1] \subseteq \mathbb{R}$ denote the closed interval between 0 and 1 on the real number line. For each $i \in [0, 1]$, let Z_i be a copy of \mathbb{Z}^- , and let $P = [0, 1] \sqcup \bigsqcup_{i \in [0, 1]} Z_i$. Define a relation \leq_P on all elements $p, q \in P$ by $p \leq_P q$ if and only if either $p, q \in Z_i$ for some $i \in [0, 1]$ and $p \leq q$, or $p \in [0, 1]$ and $q \in Z_p$. Then, as in the previous example, (P, \leq_P) is a partially ordered set. Also, P satisfies DD since every element of P is the greatest lower bound of some Z_i or subset thereof, and $\mathcal{R}(P) = P \setminus [0, 1]$. However, P clearly does not satisfy KAP, since $[0, 1] \subseteq P$. \square

The two previous examples notwithstanding, there is however a connection between the conditions KAP and DD for partially ordered sets satisfying GLB and a strengthened version of DC. (Note that the partially ordered sets in both of those examples satisfy GLB.)

Proposition 38. *Let (P, \leq) be a partially ordered set satisfying GLB, and suppose that for every downward directed $S \subseteq P$ and every $p \in P$ satisfying $p > \text{glb}(S)$, we have $p \geq s$ for some $s \in S$. Then P satisfies KAP if and only if P satisfies DD.*

Proof. Suppose that P satisfies KAP. Let $S \subseteq P$ be downward directed without least element, and set $\text{glb}(S) = p$. We wish to find a downward directed subset of $\mathcal{R}(P)$ whose greatest lower bound is p . Upon replacing S with a larger set, by Lemma 23 we may assume that if $t \in P$ and $s \in S$ are such that $s \leq t$, then $t \in S$. We shall show that $S \cap \mathcal{R}(P)$ is downward directed and has p as the greatest lower bound.

Let $q \in S$ be any element. Then, by Lemma 23(2), there exists $q' \in S$ such that $q' < q$. Since P satisfies KAP, there exist $q_1, q_2 \in P$ such that $q' \leq q_1 < q_2 \leq q$ and there are no elements of P strictly between q_1 and q_2 . We claim that $q_1 \in \mathcal{R}(P)$. If not, then there exists $T \subseteq P$ downward directed without least element, such that $\text{glb}(T) = q_1$. Again applying

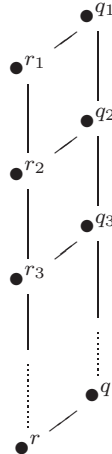
Lemma 23 we may assume that if $t \in P$ and $s \in T$ are such that $s \leq t$, then $t \in T$. By our hypothesis on P , since $q_1 < q_2$, there exists $t \in T$ such that $t \leq q_2$, and hence $q_2 \in T$. But then, by Lemma 23(2), there exists $s \in T$ such that $q_1 < s < q_2$, contradicting our choice of q_1 and q_2 . Thus $q_1 \in \mathcal{R}(P)$, and since $q' \leq q_1$, it follows that $q_1 \in S$. We have therefore shown that for every $q \in S$ there exists $q_1 \in S \cap \mathcal{R}(P)$ such that $q_1 < q$.

Next, we show that $S \cap \mathcal{R}(P)$ is downward directed. Given any $s, t \in S \cap \mathcal{R}(P)$ there must be some $r \in S$ such that $r \leq s$ and $r \leq t$, and hence there must be some $r_1 \in S \cap \mathcal{R}(P)$ such that $r_1 < r$, and therefore also $r_1 \leq s$ and $r_1 \leq t$. Finally, we note that $\text{glb}(S \cap \mathcal{R}(P)) = p$. For $S \cap \mathcal{R}(P) \subseteq S$ implies that $p \leq \text{glb}(S \cap \mathcal{R}(P))$. On the other hand if $p \neq \text{glb}(S \cap \mathcal{R}(P))$, then our assumption on P implies that $q \leq \text{glb}(S \cap \mathcal{R}(P))$ for some $q \in S$. But then $q_1 < q$ for some $q_1 \in S \cap \mathcal{R}(P)$, and hence $q_1 < \text{glb}(S \cap \mathcal{R}(P))$, which is absurd. Therefore $\text{glb}(S \cap \mathcal{R}(P)) = p$.

Conversely, suppose that P satisfies DD. Let $p, q \in P$ be such that $p < q$. We wish to find $p', q' \in P$ such that $p \leq p' < q' \leq q$, and there are no elements of P strictly between p' and q' . First, suppose that $p \in \mathcal{R}(P)$. Let S be the set of all chains of elements $s \in P$ such that $p < s \leq q$. The union of any chain of chains in S is again a chain in S , and hence, by Zorn's lemma there is a maximal chain $C \in S$. Since $C \subseteq P$ is downward directed, $p \notin C$, and $p \in \mathcal{R}(P)$, we have $p \neq \text{glb}(C)$, by Lemma 29. Also, by the maximality of C , there cannot be an element of P strictly between p and $\text{glb}(C)$. Hence, letting $p = p'$ and $\text{glb}(C) = q'$, we have the desired conclusion.

We may therefore assume that $p \notin \mathcal{R}(P)$. Since P satisfies DD, there exists a downward directed $S \subseteq \mathcal{R}(P)$ without least element, such that $\text{glb}(S) = p$. By our hypothesis on P , since $p < q$, there exists $p' \in S$ such that $p' \leq q$. Moreover, by Lemma 23(2), we may assume that $p' < q$. Since $p' \in \mathcal{R}(P)$, by the argument in the previous paragraph, there exists $q' \in P$ such that $p' < q' \leq q$ and there are no elements of P strictly between p' and q' , again giving the desired conclusion. \square

Remark 39. A moment's reflection yields that DC is equivalent to a modified version of DC, in which the requirement that $p \geq \text{glb}(S)$ is replaced by $p > \text{glb}(S)$. Thus the second hypothesis on P in Proposition 38 is almost identical to condition DC; the only difference is that $p \in \mathcal{R}(P)$ in DC, while $p \in P$ in the Proposition. So the condition of the Proposition is stronger than DC. Indeed, consider the poset P having the following Hasse diagram.



If $S_r = \{r_i \mid i \in \mathbb{Z}^+\}$ and $S_q = \{q_i \mid i \in \mathbb{Z}^+\}$, then $r = \text{glb}(S_r)$ and $q = \text{glb}(S_q)$. We see that $\mathcal{R}(P) = S_r \cup S_q$, so that every $p \in \mathcal{R}(P)$ indeed has the property specified in condition DC. However, if we consider $q \in P$, then $q > \text{glb}(S_r)$, but there is no $r_i \in S_r$ for which $q \geq r_i$. So P does not satisfy the second hypothesis of Proposition 38. \square

We have, by Proposition 3, that $\text{Spec}(R)$ satisfies GLB for any ring R . Furthermore, it is clear that any totally ordered set P satisfies the second hypothesis of Proposition 38. So we get the following.

Corollary 40. *Let R be a ring for which $\text{Spec}(R)$ is totally ordered. Then $\text{Spec}(R)$ satisfies condition KAP if and only if $\text{Spec}(R)$ satisfies condition DD.*

The difficulty in determining an answer to Question 34 persists even when a key additional assumption is made on the underlying spectra.

Question 41. *Is there a ring R such that $(\text{Spec}(R), \subseteq)$ is totally ordered, and for which $(\text{Spec}(R), \subseteq)$ does not satisfy KAP?*

Equivalently, by Corollary 40: Is there a ring R such that $(\text{Spec}(R), \subseteq)$ is totally ordered, and for which $(\text{Spec}(R), \subseteq)$ does not satisfy DD?

We conclude the article with a final question regarding the prime spectrum of a Leavitt path algebra.

Question 42. *Let (P, \leq) be a partially ordered set such that $(P, \leq) \cong (\text{Spec}(L_K(E)), \subseteq)$ for some field K and graph E . Does there necessarily exist a graph F , where F is acyclic and contains no breaking vertices, for which $(P, \leq) \cong (\text{Spec}(L_K(F)), \subseteq)$?*

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